

ITR-submission

Abhiram Srinivas

September 2025

1 An into to the title

*-IISER **-IISc *ITR stands for Imagination to reality, well it should stand for imagination to reality with the help of category theory and categorical thinking but that is way to long to be a name . in this project i aim to build a categorial intuition in the mind og the reader which by the end of hopefully will be thinking like a categoriest or atleast i hope to invoke an intrest in the reader to persue this line of thinking further.

I will state my ideas in a systemotic way but the reader will have to break thair head (when needed) to really go into the depth of the matter .

saying all that i will end by saying that i will be using the yoneda lemma readily and it would be good for the readeer to be fairly used to the definigition anf usage of the yoneda lemma , this will help me too as one of my favrete categoists Emily Riehl says this lemma is one which is decieveing and shows itself to only its most loved users , well thats not exacly what she said but it has the spirit i will alsi be trying to so some small extentions to the yoneda lemma as we go .

2 Yoneda lemma

lets state the yoneda lemma

let F, G be two parellel fucntors from C to the categorie set with $F=C(x_0, -)$ where x_0 is some elemnet of C , then

$$y : Nat(F, G) \longrightarrow G(X_0) \quad (1)$$

is a bijection

well tho the trained eye this is more than gold this is like the firt glims of sunlight to man trapped in a dark cave for a week but ofc this is my openion on the matter . i will provide a visaul flow of what is happing in the yoneda lemma as well and i will be returing to the diagram often

$$\begin{array}{ccc}
 X_1 & \xrightarrow{F} & c(X_0, X_1) \\
 & \downarrow \phi & \\
 & G(X_1) & \\
 & \searrow y(\phi)_{X_0} & \\
 & & G(X_0)
 \end{array}$$

well i would like to state that ive used the standerd textbook(marko Grandis) notations
but i will refresh the memories once more

$C \rightarrow$ category
 $X_0 \rightarrow$ object of the category
 $c(X_i, X_j) \rightarrow$ hom set of the elemts mentioned
 $G(X_i) \rightarrow$ image of the object trough the functor G
 $\phi \rightarrow$ natural transformation between the functors
 $y(\phi)_{X_0} \rightarrow$ yoneda bijection mentiond above

great , now that we have gotten that out of the way we can finally begin to write
this sond , draw the picture , hit the dance whatever artistic equevalent of starting a
perfermece floates your boat

2.1 playing around with this tool

here i go into some of the intresting rabbit holes that one can think off now that we are
equipped with the tool Yoneda lemma and see and hopefully discover something new
or even more fundamental about the lemma itself

2.1.1 Malti morphism yoneda lemma

so lets start by stating this topic more mathamatically , let us see what happens when
we tweak with the yoneda lemma a bit and say that both the parralel functors are of the
same type

$$F = c(X_0, -) \quad (2)$$

and

$$G = (X_1, -) \quad (3)$$

now on this wpecial case let us put the yonenda magic and see what the hom sets show
us

For F

$$\begin{array}{ccc}
 X_1 & \xrightarrow{F} & c(X_0, X_1) \\
 & & \downarrow \phi \\
 & & c(X_1, X_1) \\
 & & \searrow y(\phi)_{X_0} \\
 & & c(X_1, X_0)
 \end{array}$$

and now for G

$$\begin{array}{ccc}
 X_0 & \xrightarrow{G} & c(X_1, X_0) \\
 & & \downarrow \phi' \\
 & & c(X_0, X_0) \\
 & & \searrow y(\phi')_{X_1} \\
 & & c(X_0, X_1)
 \end{array}$$

and now as the viewer sees yes the sets are infact kind of the same , in fact let us put them next to each other to see

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{G} & c(X_1, X_0) & & c(X_0, X_1) \xleftarrow{F} X_1 \\
 & & \downarrow \phi' & & \downarrow \phi \\
 & & c(X_0, X_0) & & c(X_1, X_1) \\
 & & \searrow z(\phi')_{X_0} & & \swarrow y(\phi)_{X_1} \\
 & & c(X_0, X_1) & & c(X_1, X_0)
 \end{array}$$

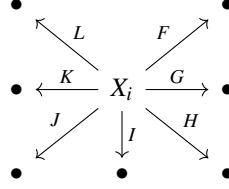
and now we can get rid of the repeting hom sets and make the diagram one big diagram instead of 2 as such

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{G} & c(X_1, X_0) & & c(X_0, X_1) \xleftarrow{F} X_1 \\
 & & \downarrow \phi' & \swarrow z(\phi')_{X_0} & \searrow y(\phi)_{X_1} \\
 & & c(X_0, X_0) & & c(X_1, X_1) \\
 & & & \swarrow y(\phi)_{X_1} & \downarrow \phi
 \end{array}$$

and im sure you see why this is awesome ! it states that the 4 hom sets have invisible strings that connect each of them ,i.e changing even on of the hom set will result in the change in the others , to be preicse the set $c(x_0, x_0)$ is ment more to be the image of phi which in the most genaral case will be some subset of the $c(x_0, x_0)$ but please keep this in mind when we discuss further as writing the subsets become tedious

now this discorse clearly has so meny ways to develum further

1.**a more general view** so in the hyper specific case above we looked at the functors for X_0, X_1 but we can actually do this for every object that is in a way



where F,G are the same as discussed before but now the H,I,J,K... are all basically the same as F and G but with different Basis

$$H = c(X_3, -), I = c(X_4, -0), J = c(X_5, -), \dots \quad (4)$$

let us see where this can take us. To start let us start with the previous diagram itself

$$\begin{array}{ccccc} X_0 & \xrightarrow{G} & c(X_1, X_0) & & c(X_0, X_1) \xleftarrow{F} X_1 \\ & & \downarrow \phi & \swarrow z(\phi')_{X_0} & \searrow y(\phi)_{X_1} \\ & & c(X_0, X_0) & & c(X_1, X_1) \\ & & & \nwarrow w(\phi'')_{X_1} & \downarrow \phi' \end{array}$$

now let us do this for the pair (x1,X2)

$$\begin{array}{ccccc} X_1 & \xrightarrow{H} & c(X_2, X_1) & & c(X_1, X_2) \xleftarrow{G} X_2 \\ & & \downarrow \phi'' & \swarrow x(\phi''')_{X_2} & \searrow \phi''' \\ & & c(X_1, X_1) & & c(X_2, X_2) \\ & & & \nwarrow w(\phi'')_{X_1} & \downarrow \phi''' \end{array}$$

now as we can see we can join these two diagrams

$$\begin{array}{ccccccc} X_0 & \longrightarrow & c(X_1, X_0) & & c(X_0, X_1) \xleftarrow{F} X_1 & \xrightarrow{H} & c(X_2, X_1) \\ & & \downarrow & \swarrow y & \searrow \phi' & & \downarrow \phi'' \\ & & c(X_0, X_0) & & c(X_1, X_1) & & c(X_1, X_1) \\ & & & \nwarrow z & \swarrow w & & \nwarrow w \\ & & & & & & c(X_1, X_2) \xleftarrow{G} X_2 \\ & & & & & & \downarrow \phi''' \\ & & & & & & c(X_2, X_2) \end{array}$$

well that was quite the squeeze , but we all see the point . i hope the short notations of the yoneda maps are clear ive refrained from using the whole expansions let me draw the general picture first before going into anything

$$\begin{array}{ccccc} & & c(X_{i-1}, X_i) \xleftarrow{F_{i-1}} X_i \xrightarrow{F_{i+1}} c(X_{i+1}, X_i) & & \\ & \swarrow & \downarrow & \searrow & \\ & c(X_i, X_i) & & c(X_i, X_i) & \end{array}$$

hopefully the object and functor indices are understood if not then

$$F_i = c(X_i, -) \text{ where } X_i \text{ is an object in the category} \quad (5)$$

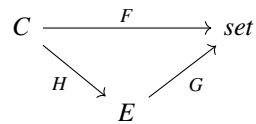
this is element of this analysis , and we just to visualize this let us represent this as 2 perpendicular lines , but im sure anyone who is following will reach a conclusion that such a procefade done to every elemet will result in every hom set that can exist and the massive diagram will be a cylindrical diagram

there is a simple way to visualize what we have just gone over as right angles IMAGE !, IMAGE 2

one last thing we can talk about is phi is a bijection or an isomorphism then we get a transfer of information without any loss which essentially means that each one of these hom sets are frames that are vuieng the same bit of information with diffrent tints

2.1.2 composition yoneda(A Spiderweb)

so now in the next section we will be talkign of the composton yoneda . what i mean by this will be clear from this diagram



here we again meet the same F but now there are two new functors H,G

$$H : C \longrightarrow E \text{ is a functor} \quad (6)$$

$$G : E \longrightarrow SET \text{ such that } G = c(Y_0, -) \quad (7)$$

where Y_0 is a object in E and is $H(x_0)$. so in a sense whe have this

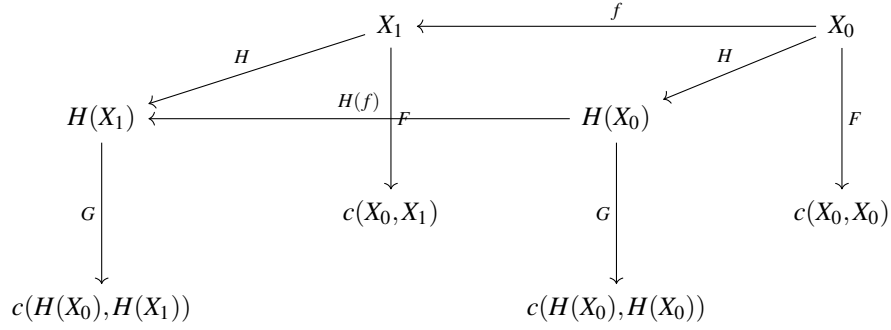
$$\begin{array}{ccc} C & \xrightarrow{F} & set \\ & \xrightarrow{G \circ H} & \end{array}$$

to this let us now put the yoneda lemma ont this to get a an important result and once we get that we can move on to see the local structure of this composition yoneda , youll see what i mean

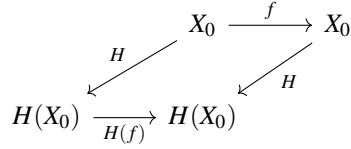
let me first draw what is happanign

$$\begin{array}{ccc} & & X_0 \\ & \swarrow H & \downarrow F \\ H(X_0) & & c(X_0, X_0) \\ \downarrow G & & \\ c(H(X_0), H(X_0)) & & \end{array}$$

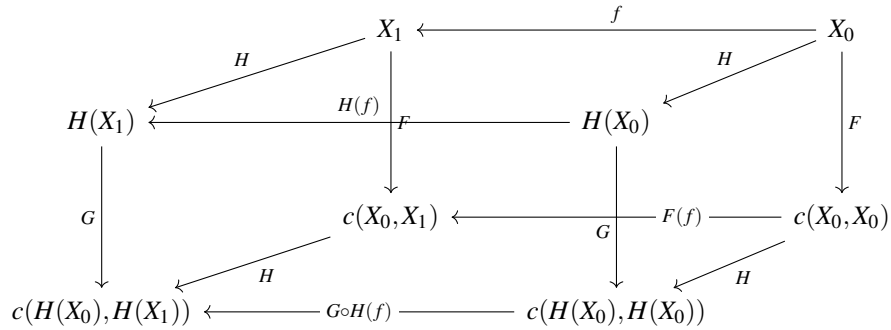
now let us look at a whole picture , cubic diagrams which represent the motion of the objects and thair morphisms essentially how the whole categorie behaves under some transformation



and as we can absurve the bottom plane is not fully connected to each other let us see the morphisms that connect them since all the hom sets that we come acros here are part of the same categorie which is set , lets start with this



and from this it is clear to see the following

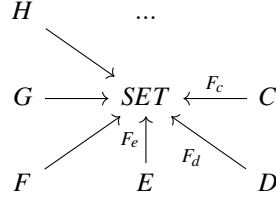


the side face connects on the bottom are just the inietal functors that were chosen , which gives us a fentastic connecting result

$$G \circ H \cong H \circ F \quad (8)$$

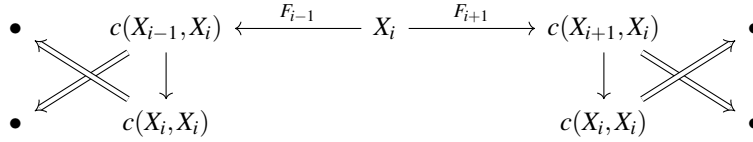
this is the glue that will now use to stick meny local malti yoneda layors together to form a global spiderweb

lets talk about what i mean by spider web , as usual this picture will depict what i mean

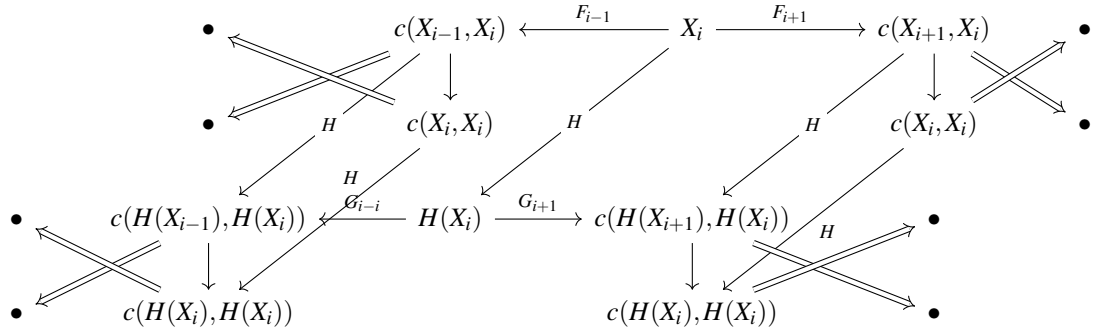


this is similar to a spider web where the centre of it is the categori of set and sarrounding it are diffrent catgroes that are attached to the centre by the yoneda functors (hom set functors) . let us now see how this looks when take a deepeer peep and see how individual malti yoneda lemmas are connected to each other

lets start with a reminder what do i mean by a local section og the malti yoneda lemmas well essentially this dagrahm



this is the local section of the entire yoneda lemma system of a perticular categori and now



as we can see this is just joingi the two local forms of the malti yoneda lemmas but as we can see these are only joint at the vortexes(hom sets) are there any joints on the bijections themselves (Double lined arrows) ? mathamatically we are asking if

$$if \exists a J : y_C \longrightarrow y_D \quad (9)$$

where

y_C is a yoneda bijection in the C categorie

y_D is a yoneda bijection in the D categorie

let us begin by writing down the two yoneda lemma from the two categories

$$y_C(\phi)_{X_0} : c(X_0, X_0) \longrightarrow c(X_0, X_1) \quad (10)$$

similarly

$$y_D(\psi)_{H(X_0)} : c(H(X_0), H(X_0)) \longrightarrow c(H(X_0), H(X_1)) \quad (11)$$

[i have referd to the huge diagram that almost doesnt fit the page please chweck that out if the equations dont make dierct sense]

so from this it is clear and we can make the comuting diagram

$$\begin{array}{ccc} c(X_0, X_0) & \xRightarrow{y_C(\phi)_{X_0}} & c(X_0, X_1) \\ \downarrow H & & \downarrow H \\ c(H(X_0), H(X_0)) & \xRightarrow{y_D(\psi)_{H(X_0)}} & c(H(X_0), H(X_1)) \end{array}$$

and we can from this clearly see

$$H \circ y_C \cong y_D \circ H \quad (12)$$

and ther is no J that that behaves in that form or atleast no trivial J , with further investigation we may find a J in another location of analysis . we will go deeper intp that wehn we discuss the happenigns in the Yoneda categorie

so we have now seen with sopme certianity that a trivial mappign from the diferntt yoneda bijections dont exist let us go into the last bit of this sction which is like a slight genarilazation of the yoneda lemma involving multiple compositions . **multiple compostite yoneda lemma**. let me again show you what i mean by this with the help of diagrams

$$C \begin{array}{c} \xrightarrow{A} \\ \xrightarrow{B} \end{array} D$$

say that

say C,D are two categorie
and A,B are 2 parallel functors from C to D

now with these 2 i want to now checj the yoneda property in a sense i want to genralize the yoneda lemma by removing the nessiscity of D being SET . a good place to bigin such analysis is this

$$\begin{array}{ccc} C & \begin{array}{c} \xrightarrow{A} \\ \downarrow \phi \\ \xrightarrow{B} \end{array} & D \\ & \begin{array}{cc} \searrow F & \swarrow G \\ & SET \end{array} & \end{array}$$

the same nomenclature applies but this time with a slight change

$F=c(X_0, -)$ for some X_0 in C
 $G=c(Y_0, -)$ for some Y_0 in D
 ϕ is some natural transformation between the parallel functors A, B

now let us write down the two yoneda lemmas

$$y_A : NAT(F, G \circ A) \longrightarrow G \circ A(X_0) \quad (13)$$

$$y_B : NAT(F, G \circ B) \longrightarrow G \circ B(X_0) \quad (14)$$

this will become

$$y_A : NAT(F, G \circ A) \longrightarrow c(Y_0, A(X_0)) \quad (15)$$

$$y_B : NAT(F, G \circ B) \longrightarrow c(Y_0, B(X_0)) \quad (16)$$

and now let us try to find a map between the two yoneda lemmas but we will do so with the help of a diagram for clearly seeing what is happenign

$$\begin{array}{ccccc}
 X_0 & \xrightarrow{B} & B(X_0) & \xrightarrow{G} & c(Y_0, B(X_0)) \\
 A \downarrow & \nearrow \phi & & \nearrow G(\phi) & \\
 A(X_0) & & & & \\
 G \downarrow & & & & \\
 c(Y_0, A(X_0)) & & & &
 \end{array}$$

now it is clearly evident that

$$G(\phi) : y_A \longrightarrow y_B \quad (17)$$

which is now a trasformation between yoneda bijections , we will dwell more into this on our section on yoneda categories and see i this is fully correct or if we are jumping to conclutions here. with this we can put and end to this section and move on to our next one which will be the last puzzle piece to our categorie of yoneda lemmas

2.1.3 yoneda backwrads

what does yoneda backwards mean , im sure the intrested reader would have figured it out . it is basically startng out at a bijection and climbing back up to the hom sets , but for that let us ask the first question ”where does the yoneda bijection stay”. lets use this set up

$$\begin{array}{ccc}
 X_i & \xrightarrow{F} & F(X_i) \\
 & & \downarrow (\phi)_{X_i} \\
 & & G(X_i) \\
 & & \searrow y(\phi)_{X_i} \\
 & & G(X_0)
 \end{array}$$

the usual definitions hold $F=c(X_0,-)$ now let's look at the inverse flow

$$y'(x)_{X_i} : F(X_i) \longrightarrow G(X_i) \quad (18)$$

let's start with this, where y' is the inverse of y . as we know $F(x_i), G(x_i)$ are part of SET so the mapping (y) between them is a morphism between 2 elements in SET

$$y'(x)_{X_i} \in c(F(X_i), G(X_i)) \quad (19)$$

now that we got where y' lives let's see where y lives

$$y(\phi)_{X_i} : G(X_i) \longrightarrow G(X_0) \quad (20)$$

so as we can see the home of our local yoneda bijection is

$$y(\phi)_{X_i} \in c(G(X_i), G(X_0)) \quad (21)$$

ok now we can go ahead and talk about the significance of these locations and morphisms

first let's state that the " y " is a bijection that is for every " y " there is a " Y " such that they are inverses, now we can ask at what level are the bijections

$$y(\phi) = x \quad (22)$$

and

$$y'(x) = \phi \quad (23)$$

here is the bijection, if there is a y there will be a y' such that they are inverses of each other, now let us see how and what this means

1. First thing we can say is that the existence of an element of an object in the image of C on G has to do with the existence of a natural transformation, seemingly unrelated things - but this is nothing new this is just the yoneda lemma

2. let us see how deep this way of thinking can take us, let us first start with an object of C then let us take the image of that with the functor G and say that is $G(X_0)$ in SET. this simple procedure now implies that there is a whole set of natural transformations between the G and F

3. this basically gives us an amazing tool, that is whenever we need to analyze natural transformations all we have to do is take the image and analyze the system through image dimensions rather than breaking our heads on all possible natural transformations etc and vice versa

4. it gives an amazing way to analyze the same info in 2 different ways

now let us move on to something more interesting, category of yoneda lemmas.

2.1.4 Yoneda categories

lets approach this through the categorie of parallel fucntors between two categories , to start let us defignt he 2 categories
the end goal of this section is to have a nice way to relate yoneda lemmas in n categories to the basic yoneda lemma , and maybe take some large n limits and so on

$$C \begin{array}{c} \xrightarrow{F} \\ \xrightarrow{G} \\ \xrightarrow{H} \end{array} SET$$

and so on where the obv definitions still hold but with a slight generilaization to begin with

$$F=c(X_0, -) \\ G, H, \dots \text{ some functors}$$

ok so now let us defign the categorie of all parrallel fucntors betwenn teh two categories and lets call it $\mathbf{F}$

$$\mathbf{F} \text{ is the categorie of all parrallel functors} \quad (24)$$

suche that

$$ob(\mathbf{F}) = [F, G, H, \dots] \quad (25)$$

and the morphims sets are just the natural transformations between the functors

$$HOM(\mathbf{F}) = [NAT(F, G), NAT(F, H), \dots] \quad (26)$$

so now let us defign another fucntor

$$Y : \mathbf{F} \longrightarrow SET \quad (27)$$

such that $Y=c(F, -)$ just the same procedure on how we defign the yoneda lemma we are all used to

now here i must add that the following next steps we take are going to be big steps so the reader might need to pause whenever nessesort and things do get messy for a while from now, saying that let us bigin

$$A, B : \mathbf{F} \longrightarrow SET \quad (28)$$

be two parrallel functors such that

$$A=c(F, -) \text{ Where } F \text{ is a object of the categorie and } c \text{ is the hom set} \\ B = \text{ some functor to } SET$$

so now the yoneda lemma will be

$$t : NAT(A, B) \longrightarrow B(F) \quad (29)$$

is a bijection now this is contains a boatload of information to process , and since we will need it let me just mention the classic yoneda lemma once

$$y : NAT(F, G) \longrightarrow G(X_0) \quad (30)$$

is a bijection

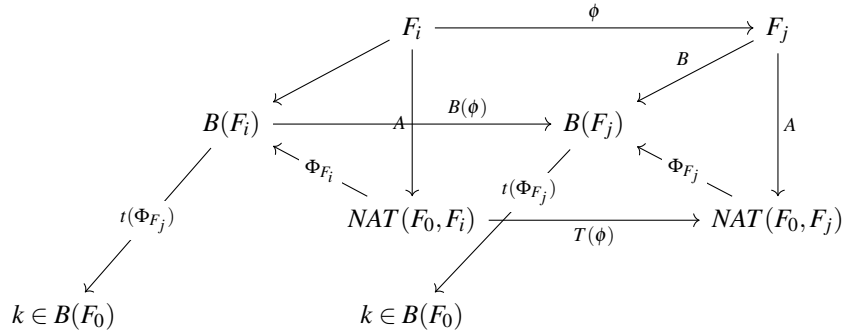
now lets see in how many places our classic yoneda lemma hides in this yoneda lemma , and keep in mind in this whole exploration we need to lkeep in mind the procedure of investigation becuse we can again go into the 3rd degree category (3rd dimentions) and do this procedure again there and so on

but first a quick redefinition of our categorie of funcotrs for smoother analysis

$$ob(\mathbf{F}) = [F_0, F_1, F_2, \dots] \quad (31)$$

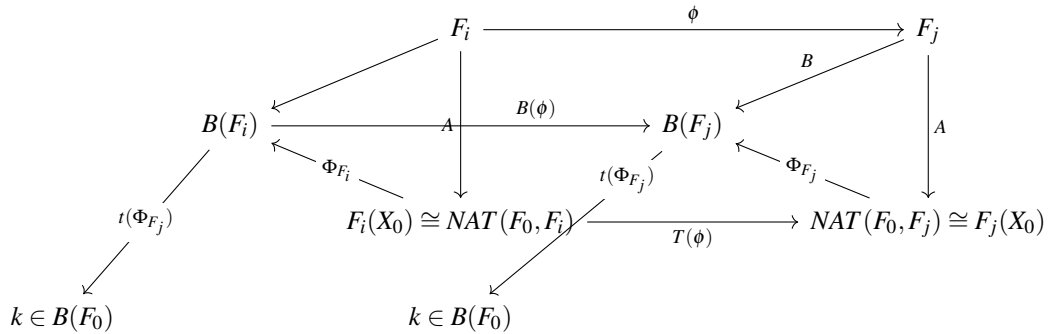
and say for refrence sake $F_0 = c(X_0, -)$

now we can start to see what happens



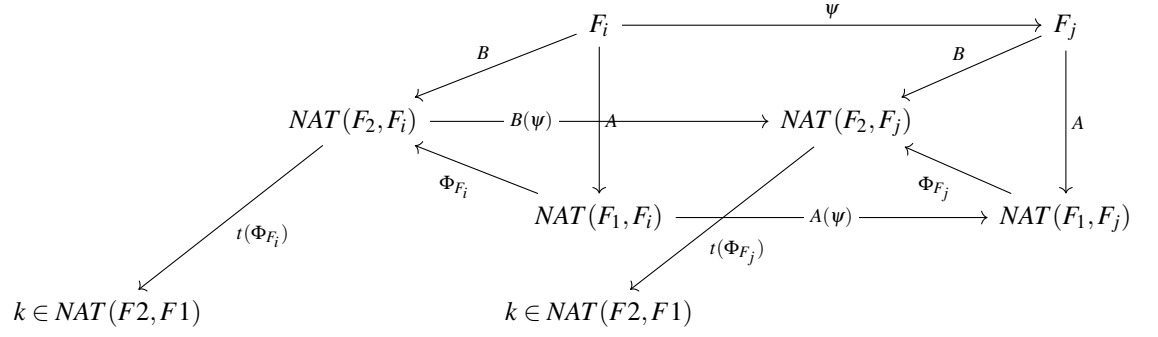
i hope this is a clear dipection of the yoneda lemma of the dim 2 categorie

now i hope we can all see where in this picture our basic yoenda lemma comes into play



it will come at the natural trasformation places, and a point to note is that we have to use 2 separte yoneda lemmas here for this dim 2 analyissi , i can easily conjecture that every next step will use 2 previus yoneda lemma

let us now zoom into a bit of specialization in this analysis and state that every parallel functor A,B,.. will be of the form $A_i = c(F_i, -)$ so lets rewrite our diagram as such d



where clearly

$$\begin{aligned} A &= c(F_1, -) \\ B &= c(F_2, -) \end{aligned}$$

from this we can clearly see from all this the basic yoneda lemma embedded into this