

How does n Yoneda behave?

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1 Introduction

This is a procedure to generalize the Yoneda lemma to higher dimensions using the standard Yoneda lemma (1-Yoneda lemma) as a parameter of sorts.

The flow of the project will go as such:

We start by defining 2-categories and then n-categories. This is followed by the 2-Yoneda lemma and how all these higher Yoneda lemmas are linked to the 1-Yoneda lemma. Finally, we conclude with where we would want to use this tool.

2 Higher Categories

Say there exist two categories C and D and parallel functors F, G, H and I such that

$$\begin{array}{ccc} & \xrightarrow{F} & \\ C & \xrightarrow{G} & D \\ & \xrightarrow{H} & \\ & \xrightarrow{I} & \end{array}$$

From this we can see the appearance of a category of functors with the morphism class only being the natural transformations (represented by the double arrows).

$$\begin{array}{ccc} & \xrightarrow{F} & \\ C & \begin{array}{c} \Downarrow \\ \Downarrow \\ \Downarrow \end{array} \xrightarrow{G} & \begin{array}{c} \Downarrow \\ \Downarrow \\ \Downarrow \end{array} D \\ & \xrightarrow{H} & \end{array}$$

This is what we might call 2-categories. However, this procedure can be carried out again by replacing C with the 2-category just taken and having another set of parallel functors from the category of functors to another category.

Let us be more specific by changing D to SET

$$\begin{array}{ccc} & \xrightarrow{F} & \\ C & \begin{array}{c} \Downarrow \\ \Downarrow \\ \Downarrow \end{array} \xrightarrow{G} & \begin{array}{c} \Downarrow \\ \Downarrow \\ \Downarrow \end{array} SET \\ & \xrightarrow{H} & \end{array}$$

Let us formalize the notion of 2-categories and apply the Yoneda lemma on this setup.

Formalizing 2-categories

$$\mathbf{F} \Longrightarrow ob\mathbf{F} = [F, G, H, \dots] \quad (1)$$

and the morphism set will be the natural transformation between them

$$hom(F, G) \cong NAT(F, G) \quad (2)$$

The Yoneda lemma for this will be defined as such:

Let A,B be 2 parallel functors such that

$$A, B: \mathbf{F} \longrightarrow SET$$

such that for any object F of the 2-categories

$$A \longrightarrow hom(F, -)$$

then

$$y_2 : NAT(A, B) \longrightarrow B(F) \quad (3)$$

is a bijection. This will be the Yoneda lemma for the second order. Let us now see the direct consequences of this and the existence of the first order Yoneda lemma on this.

$$\begin{array}{ccccc}
 & & H & \xrightarrow{\quad \phi \quad} & G \\
 & \swarrow B & \downarrow A & & \swarrow B \\
 B(H) & & & & B(G) \\
 \swarrow y_2 & \nwarrow \Phi_H & \downarrow A & \swarrow y_2 & \nwarrow \Phi_G \\
 I \in B(F) & & H(X_0) & & NAT(F, G) \\
 & & \downarrow A & & \\
 & & I \in B(F) & &
 \end{array}$$

where

$$\Phi_G : A(G) \longrightarrow B(G) \quad (4)$$

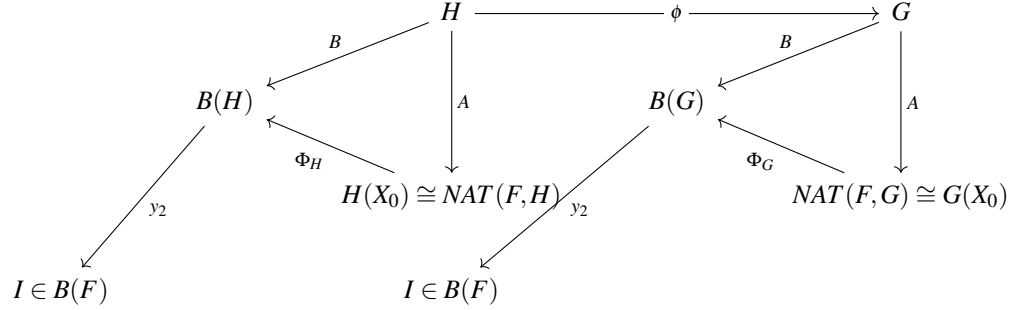
Restricting F to

$$F = c(X_0, -)$$

we see the 1-Yoneda connection for any object (of $\mathbf{F}$) G is as such

$$NAT(F, G) \cong G(X_0) \quad (5)$$

As we can see, there is a "parametrization" of the 1-Yoneda in the 2-Yoneda's logic flow.



Observing this, let us go deeper into 2-Yoneda and try to answer if we can have 1-Yoneda like behavior without the hom functor condition.

Restricting B as a hom functor, we see

$$y_2 : NAT(hom(F, -), hom(G, -)) \longrightarrow hom(G, F) \quad (6)$$

is a bijection

$$y_2(\Phi) \longrightarrow I \in hom(F, G) \quad (7)$$

and

$$y_2^{-1}(I) = \Phi \quad (8)$$

so we can see that

$$y_2^{-1}(I)_H = \Phi_H : hom(F, H) \longrightarrow hom(G, H) \quad (9)$$

but from the 1-Yoneda lemma, we can say

$$y_2^{-1}(I)_H = \Phi_H : hom(F, H) \cong H(X_0) \longrightarrow hom(G, H) \quad (10)$$

introducing y_1 (1-Yoneda bijection)

$$y_2^{-1}(I)_H \circ y_1^{-1} : H(X_0) \longrightarrow hom(G, H) \quad (11)$$

From (11) we can say if the Φ_H component of the natural transformation is an isomorphism, then we can have an equivalence between $H(X_0)$ and $NAT(G, H)$.

G and H are 2 parallel functors from C to SET, not necessarily hom functors. The point to drive home is that due to this isomorphism, there is a Yoneda-like behavior without ever bringing in hom functors.

The condition for the existence of this wouldn't have been so clear had we not looked

at the problem from a higher position; this is what we are motivating.

There are some more relations we can form.

Let's first start off at the 2-Yoneda lemma,

$$y_2 : NAT(A, B) \longrightarrow B(F) \quad (12)$$

continuing under the same conditions with an addition as such:

say that B is $\text{hom}(G, -)$ for some G in $\mathbf{F}$ such that $G = c(X_i, -)$
and A is $\text{hom}(F, -)$ as discussed

We see

$$y_2 : NAT(\text{hom}(F, -), \text{hom}(G, -)) \longrightarrow \text{hom}(G, F) \quad (13)$$

$$y_2 : NAT(\text{hom}(F, -), \text{hom}(G, -)) \longrightarrow \text{hom}(G, F) \cong F(X_i) \quad (14)$$

Essentially, we find a bijection

$$NAT(\text{hom}(F, -), \text{hom}(G, -)) \longrightarrow F(X_i) \quad (15)$$

$$\dots \cong NAT(NAT(F, -), NAT(G, -)) \cong NAT(G, F) \cong F(X_i) \quad (16)$$

This will be a direct extension to n dimensions as long as we set our conditions accordingly.

Going back to

$$y_2 : NAT(\text{hom}(F, -), \text{hom}(G, -)) \longrightarrow \text{hom}(G, F) \quad (17)$$

We can insert the 1-Yoneda lemma much earlier in the analysis.

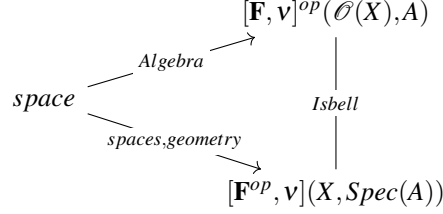
$$y_2 : NAT(\text{hom}(F, -) \cong -(X_0), \text{hom}(G, -) \cong -(X_0)) \longrightarrow \text{hom}(G, F) \cong F(X_i) \quad (18)$$

3 Future directions

3.0.1 Isbell duality

We can explore these results on enriched 2-categories and mainly on the Isbell duality to get a deeper insight into deeper, more fundamental connections between Presheaves and Copresheaves.

The goal is to find the 'space' from which both geometry and algebra arise. A diagrammatic representation would look like:



The Isbell duality relates higher geometry (Presheaves) and higher algebra (Copresheaves).

$$(\mathcal{O} \vdash spec) : Copresheaves \leftrightarrow Presheaves \quad (19)$$

Rigorously, it is represented as such, with C being any small V -enriched category:

$$(\mathcal{O} \vdash spec)_{C,v} : [C, v]^{op} \leftrightarrow [C^{op}, v] \quad (20)$$

where

$$\mathcal{O}(X)_{C,v} : c \longmapsto [C^{op}, v](X, C(-, c)) \quad (21)$$

and

$$spec(A)_{C,v} : c \longmapsto [C, v]^{op}(C(c, -), A) \quad (22)$$

Going deeper into this statement, we will start with this type of category:

$$\mathbf{F} = \mathbf{v} - \text{enriched2} - \text{category}$$

This is the category of the parallel functors between a category C and its enrichment V . We do the same procedure as above, except we replace SET with V .

This is the statement we will be going into:

$$[\mathbf{F}, v]^{op}(\mathcal{O}(X), A) := [\mathbf{F}^{op}, v](X, Spec(A)) \quad (23)$$

Taking the end expression

$$[\mathbf{F}, v]^{op}(\mathcal{O}(X), A) = \int_{F \in \mathbf{F}} v(A(F), \mathcal{O}(X)(F)) \quad (24)$$

Expanding $\mathcal{O}(X)(F)$

$$[\mathbf{F}, v]^{op}(\mathcal{O}(X), A) = \int_{F \in \mathbf{F}} v(A(F), [\mathbf{F}^{op}, v](X, c(-, F))) \quad (25)$$

Then

$$[\mathbf{F}, v]^{op}(\mathcal{O}(X), A) = \int_{F \in \mathbf{F}} \int_{G \in \mathbf{F}} v(A(F), v(X(G), c(G, F))) \quad (26)$$

Rearranging, we get

$$[\mathbf{F}, \mathbf{v}]^{op}(\mathcal{O}(X), A) = \int_{G \in \mathbf{F}} \int_{F \in \mathbf{F}} \mathbf{v}(X(G), \mathbf{v}((A(F), c(G, F)))) \quad (27)$$

From this we get

$$[\mathbf{F}, \mathbf{v}]^{op}(\mathcal{O}(X), A) = \int_{G \in \mathbf{F}} \mathbf{v}(X(G), [\mathbf{F}, \mathbf{v}]^{op}(c(G, -), (A))) \quad (28)$$

and finally

$$[\mathbf{F}, \mathbf{v}]^{op}(\mathcal{O}(X), A) = \int_{G \in \mathbf{F}} \mathbf{v}(X(G), \text{Spec}(A)(G)) \quad (29)$$

Fixing A, the transformation of Spec(A) behaves as such:
Spec(A) centered at F -

$$\text{spec}(A)(F) \longrightarrow [\mathbf{F}, \mathbf{v}]^{op}(C(F, -), A) \quad (30)$$

Taking it centered at G-(hom functor $c(X_i, -)$):

$$\text{spec}(A)(G) \longrightarrow [\mathbf{F}, \mathbf{v}]^{op}(C(G, -), A) \quad (31)$$

We bring back the functors A and B, but they are now enriched. We rewrite the above equation as such:

$$\text{spec}(A)(F) \longrightarrow [\mathbf{F}, \mathbf{v}]^{op}(A(\text{functor}), A) \quad (32)$$

and

$$\text{spec}(A)(G) \longrightarrow [\mathbf{F}, \mathbf{v}]^{op}(B, A) \quad (33)$$

Let

$$\Phi \in \text{NAT}(A, B)$$

From the above, we can state

$$\Phi : \text{Spec}(A)(F) \longrightarrow \text{Spec}(A)(G) \quad (34)$$

This should be fine as Φ acts only on A, B (functors). Φ is a natural transformation and hence has an image in $\text{NAT}(G, F)$ from 2-Yoneda lemma

$$y_2 : \text{NAT}(\text{hom}(F, -), \text{hom}(G, -)) \longrightarrow \text{hom}(G, F) \quad (35)$$

Similarly, this has an image in $F(X_i)$ from the 1-Yoneda lemma

$$y_1 : \text{NAT}(G, F) \longrightarrow F(X_i) \quad (36)$$

In conclusion, we can say

$$y_2^{-1} \circ y_1^{-1}(x) : x \in (G(x_0)) \longmapsto \Phi : \text{Spec}(A)(F) \longrightarrow \text{Spec}(A)(G) \quad (37)$$

This connection is significant because simple functor images give us insight into generalized algebra transformations. Naturally, the same can be done to generalized geometry transformations as well. The take-home statement will be

Study of Functor images
 $\cong$ **Study of a class of algebra and Geometry transformations**

Coming back to track from (29),

$$[\mathbf{F}, \nu]^{op}(\mathcal{O}(X), A) = \int_{G \in \mathbf{F}} \nu(X(G), Spec(A)(G)) \quad (38)$$

will then just be

$$[\mathbf{F}, \nu]^{op}(\mathcal{O}(X), A) := [\mathbf{F}^{op}, \nu](X, Spec(A)) \quad (39)$$

There is a lot more to go into and with a lot more rigor. These tangents from the typical derivation (of equation (20)) allow us to visit many places and form previously unknown equivalences.

3.0.2 Categorical K-theory

This is similar to K-theory - we take a class of all possible enrichments and do operations there. The base space is $\mathbf{F}$, similar to how we take the class of vector bundles and conduct operations on that class while the base space remains as is.

Starting with a category $\mathbf{C}$, let $\mathbf{k}$ be a class such that

$$\mathbf{k} = [\mathbf{SET}, \nu, \mu, , , : \text{they can form an enriched category with } \mathbf{C}]$$

We then explore the relations between different enriched categories with the help of the Isbell duality and n-Yoneda lemmas.

$$\nu - enriched \mathbf{C}$$

$$(\mathcal{O} \vdash spec)_{\mathbf{C}, \nu} : [\mathbf{C}, \nu]^{op} \leftrightarrow [\mathbf{C}^{op}, \nu] \quad (40)$$

and

$$\mu - enriched \mathbf{C}$$

$$(\mathcal{O} \vdash spec)_{\mathbf{C}, \mu} : [\mathbf{C}, \mu]^{op} \leftrightarrow [\mathbf{C}^{op}, \mu] \quad (41)$$

We can form a monoidal enriched category:

$$\nu - enriched \mu$$

$$(\mathcal{O} \vdash spec)_{\mu, \nu} : [\mu, \nu]^{op} \leftrightarrow [\mu^{op}, \nu] \quad (42)$$

and

μ – enriched v

$$(\mathcal{O} \vdash spec)_{v,\mu} : [v, \mu]^{op} \leftrightarrow [v^{op}, \mu] \quad (43)$$

One can then conjecture the existence of morphisms:

$$\pi : \mathcal{O}_{C,v} \longrightarrow \mathcal{O}_{C,\mu} \quad (44)$$

and

$$\Pi : Spec_{C,v} \longrightarrow Spec_{C,\mu} \quad (45)$$

This would then give us relations between relations of higher geometry and higher algebras.

From this, it is clear that we get a broader and deeper view of the base category from the interactions of the enriching categories analogous to how the study of vector bundles provides information about the base spaces.