

Introducing Categories to the Practicing Physicist

- Bob Coecke, arXiv (2008)

Just some pointers

- (Source) implies that it was taken from the paper [arXiv:0905.3010](#)
- This paper is to motivate the listener to get into the field , the paper itself may not hold new results etc .
- A lot of unanswered questions must be explored by the listener!

Why? (would a physicist care about category theory)

- Chris Isham's and Lee Smolin's ideas on the use of topos theory in quantum gravity
- Connections between knot theory and braided categories
- Sophisticated mathematical physics such as quantum groups and TQFT
- (For those into pure mathematics) The incredible unifying power of mathematical structures and constructions.

**A category is the exact
mathematical structure
of practicing physics!**

Why? (should a physicist care about category theory)

- Consider a physical system of type A (e.g. a qubit, or two qubits, or an electron, or classical measurement data)
- Perform an operation f on it (e.g. perform a measurement on it)
- This results in a system possibly of a different type B (e.g. the system together with classical data which encodes the measurement outcome, or, just classical data in the case that the measurement destroyed the system)

$$A \xrightarrow{f} B$$

Why? (should a physicist care about category theory)

- Next, perform an operation

$$B \xrightarrow{g} C$$

$$A \xrightarrow{g \circ f} C$$

- If we set $A \xrightarrow{1_A} A$ for the operation ‘doing nothing on a system of type A ’ we have

$$1_B \circ f = f \circ 1_A = f$$

We have a category!

What? (did we build till now)

- Well, we built a category

Definition. A *category* $\mathbf{C}$ consists of:

- *objects* $A, B, C, \dots$,
- *morphisms* $f, g, h, \dots \in \mathbf{C}(A, B)$ for each pair A, B ,
- *composition* of each $f \in \mathbf{C}(A, B)$ with each $g \in \mathbf{C}(B, C)$ resulting in $g \circ f \in \mathbf{C}(A, C)$ and this composition is such that

$$(h \circ g) \circ f = h \circ (g \circ f),$$

- *identity morphisms* $1_A \in \mathbf{C}(A, A)$ for all A which satisfy

$$f \circ 1_A = 1_B \circ f = f.$$

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Where? (is it used)

If I were to start mentioning all the fields which can be analyzed through categorical analysis we will be here forever and I can't go back and watch one piece so just to give you a few examples we have:

- Groups, isomorphisms and monoids
- Partially ordered sets
- And fine I'll fold - some quantum mechanics applications

Groups

- A Group G can be seen as a category with a single object where every morphism is an isomorphism

Definition. A morphism $f : A \rightarrow B$ is an *isomorphism* (*iso*) if it has an inverse i.e. there exists $f^{-1} : B \rightarrow A$ such that

$$f^{-1} \circ f = 1_A \quad \text{and} \quad f \circ f^{-1} = 1_B .$$

Source

- A group without inverses is called a **monoid** (very weak definition)
- (A question - think about why a monoid category will have only one object?)

[Hint: it's in the definition of a monoid and (spoiler!) it has a lot more in its definition that we will recognize and be able to use]

Partially ordered sets

- A partially ordered set P is a category with elements of this poset as objects
- Whenever $a \leq b$ we take $P(a,b)$ to be a singleton, otherwise we take it to be empty
- Closedness under composition is guaranteed by transitivity and the identities are provided by reflexivity.
- Hence, posets are examples of categories with only a few morphisms.
- A preordered set can be defined as a category in which there is at most one arrow from an object to another (in the same way as above)

Some Quantum applications

- Quantum here is just the beginning of the physical applications and if I were to go into detail about the how's , why's , what's we would be here for another 2 hours

Coecke, B. (2005) *Kindergarten quantum mechanics — lecture notes*. In: *Quantum Theory: Reconsiderations of the Foundations* III. A. Khrennikov, Ed., American Institute of Physics Press. [arXiv:quant-ph/0510032](https://arxiv.org/abs/quant-ph/0510032)

Which

- Well till now we have gone over the fundamentals
 - The structure lives in the space of operations (vs. state space),
 - Types enable to distinguish different kinds of systems,
 - Composition/application is the primitive ingredient.
- But till now just the definitions are of almost no consequence to the untrained eye , so let us try to climb down from the highest point of the abstraction mountain to a lower point

- Let us start with symmetric monoidal categories

Definition. A *symmetric monoidal category* is a category with a *symmetric monoidal tensor*, that is, an assignment both for pairs of objects and pairs morphisms

$$\begin{aligned} (A, B) &\mapsto A \otimes B \\ (A \xrightarrow{f} B, C \xrightarrow{g} D) &\mapsto A \otimes C \xrightarrow{f \otimes g} B \otimes D \end{aligned}$$

which is *bifunctorial*, and comes together with *left & right unit natural isos*, a *symmetry natural iso* and an *associativity natural iso*.

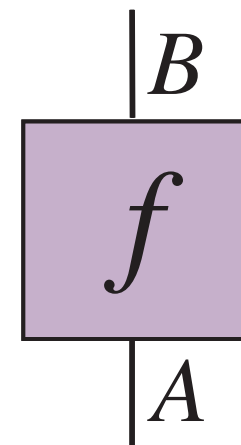
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- (Q- find how monoids and category of monoids are related)
- Lets start depicting morphism (physical processes) as little square boxes, and also label the inputs and outputs of these by “types”

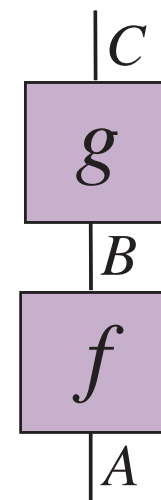
- And sequential composition is done by matching inputs and outputs of the systems
- Let us practice this nomenclature



$$1_A : A \rightarrow A$$



$$f : A \rightarrow B$$



$$g \circ f \text{ -source}$$

- And now that we have familiarized ourselves with these let's try a few examples

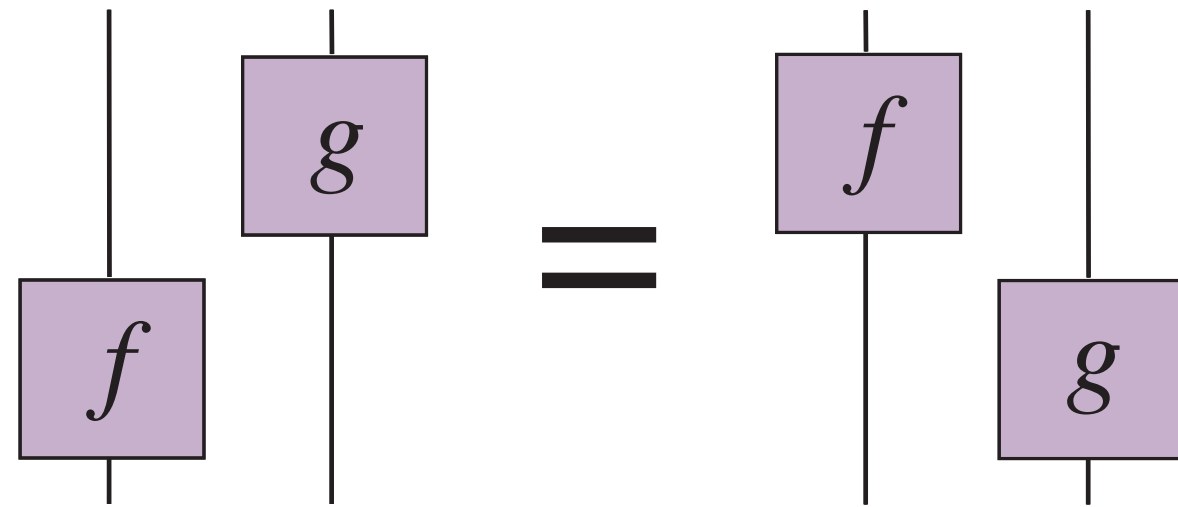
$$1_A \otimes 1_B : A \otimes B \rightarrow A \otimes B,$$

$$f \otimes 1_C,$$

$$(f \otimes g) \circ h$$

Bifunctionality

- Graphically we can represent bifunctionality like



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- Which in words just means if we apply an operation f to one system and an operation g to another system, then the order in which we apply them doesn't matter
- Hence bifunctionality expresses some notion of locality but still allows for the quantum type of non-locality

- Let us look at this mathematically also

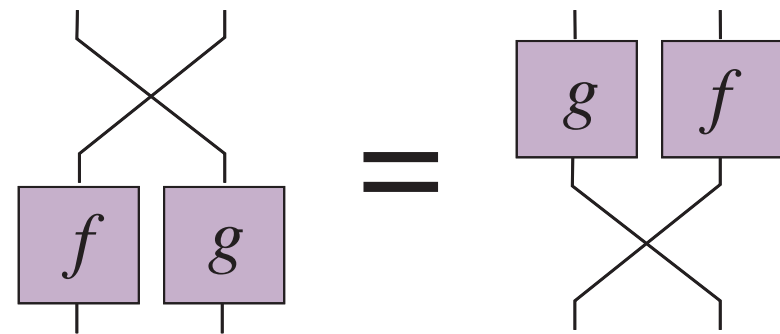
$$\begin{array}{ccc}
 A_1 \otimes A_2 & \xrightarrow{f \otimes 1_{A_2}} & B_1 \otimes A_2 \\
 \downarrow 1_{A_1} \otimes g & & \downarrow 1_{B_1} \otimes g \\
 A_1 \otimes B_2 & \xrightarrow{f \otimes 1_{B_2}} & B_1 \otimes B_2
 \end{array}$$

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- This is the mathematical reasoning of that diagram and hence we can see the how a cute diagram hid so much information about systems.

Symmetry and associativity natural isomorphisms

- We can think of natural isomorphism as a “explicitly witnessed” canonical isomorphism
- And it looks like



- Which is basically saying

$$\begin{array}{ccc} A_1 \otimes A_2 & \xrightarrow{f \otimes g} & B_1 \otimes B_2 \\ \downarrow \sigma_{A_1, A_2} & & \downarrow \sigma_{B_1, B_2} \\ A_2 \otimes A_1 & \xrightarrow{g \otimes f} & B_2 \otimes B_1 \end{array}$$

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- And we call the family of (system swapping) morphisms as the natural isomorphism

$$\{\sigma_{A,B} : A \otimes B \rightarrow B \otimes A\}$$